Variational-Iterative Method for Conductive–Radiative Heat Transfer in Spherical Inhomogeneous Medium

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Introduction

THERE are numerous applications of coupled conductive–radiative heat transfer in participating media. The heat transfer in spherical media corresponds to many important applications in different areas such as spherical propulsion systems, nuclear energy generation and explosions, astrophysics, thermal insulation systems, glass industry, and solar energy systems. Several authors have treated the problem of combined steady-state conduction and radiation heat transfer in homogeneous, spherical media for isotropic and anisotropic scattering.[1–4]

In this study, the coupled conductive–radiative heat transfer through an inhomogeneous, turbid, anisotropically scattering medium is studied. The medium is considered to be a spherical shell with diffusely reflecting and isothermal boundaries. The variational method[5,6] is used to solve the radiative transfer problem using trial functions in terms of the Pomraning[7] solution of the corresponding free-source problem. An iterative method is taken to solve the nonlinear relation between the conductive energy equation and the radiative problem. The calculations are carried out for inhomogeneous media with isotropic and forward and backward anisotropic scattering. The boundaries of the media are considered to be isothermal and may be transparent or diffusely reflecting boundaries. The calculations are used to study the effects of the single scattering albedo, the anisotropic scattering parameter, the conduction–radiation parameter, and the heat source.

Basic Equations

The dimensionless temperature distribution $\Theta(\tau)$ through a participating medium between two concentric spheres with radii τ_1 and τ_2 is described by the conservation of energy equation[3]

$$\mathrm{d}\left\{\tau^2\left[\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau}-\frac{G_1(\tau)}{2N}\right]\right\}\Big/\mathrm{d}\tau+\tau^2 H=0 \qquad (1)$$

subject to the boundary conditions

$$\Theta(\tau_1)=\Theta_1, \qquad \Theta(\tau_2)=\Theta_2 \qquad (2)$$

where τ is the optical location $(\int \mathrm{d}s\sigma_t)$, σ_t is the extinction coefficient, N is the conduction–radiation parameter $(k\sigma_t/4n^2\sigma T_r^4)$, k is the thermal conductivity, n is the refractive index of the medium, σ is the Stephan–Boltzmann constant, T_r is a reference temperature, H is used to denote prescribed heat generation in the medium $(h/k\sigma_t^2 T_r)$, h is the volumetric heat generation, Θ is the dimensionless temperature (T/T_r), and Θ_1 and Θ_2 are the dimensionless temperatures of both the inner and outer boundaries, respectively, which are constants. $G_1(\tau)$ is the dimensionless net flux $[\pi q(\tau)/n^2\sigma T_r^4$, where $q(\tau)$ is the radiative net flux] due to radiation transfer and is defined by

$$G_1(\tau)=\int_{-1}^{1}\mathrm{d}\mu\mu\Psi(\tau,\mu) \qquad (3)$$

$\Psi(\tau,\mu)$ is the dimensionless radiance $[\pi I(\tau,\mu)/n^2\sigma T_r^4$, where $I(\tau,\mu)$ is the radiative intensity] at optical position τ and scattering angle cosine μ and is described mathematically in an anisotropic scattering, inhomogeneous, spherical medium by the radiative transfer equation[8]

$$\frac{\mu\partial\Psi(\tau,\mu)}{\partial\tau}+\left(\frac{1-\mu^2}{\tau}\right)\frac{\partial\Psi(\tau,\mu)}{\partial\mu}+\Psi(\tau,\mu)=\frac{\omega(\tau)}{2}\int_{-1}^{1}\mathrm{d}\mu' p(\mu,\mu')\Psi(\tau,\mu')+[1-\omega(\tau)]\Theta^4(\tau)$$
$$-1\le\mu\le 1, \qquad \tau_1\le\tau\le\tau_2 \qquad (4)$$

with boundary conditions

$$\Psi(\tau_1,\mu)=\varepsilon_1\Theta_1^4+2\rho_1^d J^-(\tau_1)$$
$$\Psi(\tau_2,-\mu)=\varepsilon_2\Theta_2^4+2\rho_2^d J^+(\tau_2), \qquad \mu\ge 0 \qquad (5)$$

Here $p(\mu,\mu')$ is the scattering phase function, which can be given as an expansion in terms of the Legendre polynomial $P_m(\mu)$, and $\omega(\tau)$ is the spatially dependent single scattering albedo. Also, ε_i and ρ_i^d, with $i=1$ and 2, are the emissivities and diffuse reflectivities, respectively, of the boundaries and are related by $\varepsilon_i+\rho_i^d=1$. The partial radiative fluxes $J^{\pm}(x)$ are defined by

$$J^{\pm}(x)=\int_0^1\mathrm{d}\mu\mu\Psi(x,\pm\mu) \qquad (6)$$

This combined conductive–radiative heat transfer problem has mathematical complexity due to the nonlinear relation of the temperature distribution between both the radiative transfer equation and the conductive energy equation. Also, the solution of the radiative transfer equation itself shows complexity due to the need for a particular solution of the integro-differential equation due to the existence of the energy source. Using the integral equations of the problem that contains the effect of the energy source and the boundary conditions without needing the particular solution solves the latter complexity. Using an iterative technique to overcome the nonlinearity relationship of the temperature distribution solves the first complexity.

Integral Equations

The conductive energy Eq. (1) under the effect of the boundary conditions of Eq. (2) can be integrated to give

$$\Theta(\tau)=\Theta_2-\left(\frac{\tau_2-\tau}{\tau\tau_2}\right)\left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)\left\{(\Theta_2-\Theta_1)+\left(\frac{H}{6}\right)\left(\tau_2^2-\tau_1^2\right)-\left[\int_{\tau_1}^{\tau_2}\mathrm{d}\tau' G_1(\tau')\right]\Big/(2N)\right\}+\left(\frac{H}{6}\right)\left(\tau_2^2-\tau^2\right)-\left[\int_{\tau}^{\tau_2}\mathrm{d}\tau' G_1(\tau')\right]\Big/(2N) \qquad (7)$$

The integro-differential radiative transfer equation represented by Eq. (4) with the boundary conditions of Eq. (5) is transformed, for linear anisotropic scattering, into integral form to give both the dimensionless radiance $G_0(\tau)$ and the dimensionless net radiative flux $G_1(\tau)$ as[5]

$$\tau G_m(\tau)=H_m(\tau)+\frac{1}{2}\int_{\tau_1}^{\tau_2}\mathrm{d}y\omega(y)y\sum_{n=0}^{1}a_n G_n(y)L_{mn}(\tau,y)+\int_{\tau_1}^{\tau_2}\mathrm{d}yy[1-\omega(y)]\Theta^4(y)L_{m0}(\tau,y) \qquad (8)$$

where

$$G_m(x)=\int_{-1}^{1}\mathrm{d}\mu P_m(\mu)\Psi(x,\mu), \qquad m=0,1 \qquad (9a)$$

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$$H_m(x) = \left[(-1)^{m+1}/Y_1\right][Y_2 K_{1m}(\tau_1, x) - Y_3 K_{1m}(\tau_2, x)] \tag{9b}$$

$$L_{mn}(x, y) = \left[(-1)^{m+1}/Y_1\right]\left[\rho_1^d K_{1m}(\tau_1, x) V_{1n}(y) - \rho_2^d K_{1m}(\tau_2, x) U_{1n}(y)\right] + K_{mn}(x, y) \tag{9c}$$

$$V_{mn}(y) = \rho_2^d K_{11}(\tau_1, \tau_2) K_{mn}(\tau_2, y) - \left(1 + 2\rho_2^d F\right) K_{mn}(\tau_1, y) \tag{9d}$$

$$U_{mn}(y) = K_{mn}(\tau_2, y) - \rho_1^d K_{11}(\tau_1, \tau_2) K_{mn}(\tau_1, y) \tag{9e}$$

$$K_{mn}(x, y) = \int_{-(x^2-\tau_1^2)^{\frac{1}{2}} + (y^2-\tau_1^2)^{\frac{1}{2}}}^{-|x-y|} \mathrm{d}t \left[\frac{\exp(t)}{t}\right] \times P_m\left(\frac{y^2 - x^2 - t^2}{2xt}\right) P_n\left(\frac{y^2 - x^2 + t^2}{2yt}\right) \tag{9f}$$

$$Y_1 = 1 + 2\rho_2^d F - 4\rho_1^d \rho_2^d K_{11}^2(\tau_1, \tau_2) \tag{9g}$$

$$Y_2 = \varepsilon_1 \Theta_1^4 \tau_1 \left(1 + 2\rho_2^d F\right) + 2\varepsilon_2 \Theta_2^4 \tau_2 \rho_1^d K_{11}(\tau_1, \tau_2) \tag{9h}$$

$$Y_3 = \varepsilon_2 \Theta_2^4 \tau_2 + 2\varepsilon_1 \Theta_1^4 \tau_1 \rho_2^d K_{11}(\tau_1, \tau_2) \tag{9i}$$

$$F = \left\{1 - \left[1 + 2\left(\tau_2^2 - \tau_1^2\right)^{\frac{1}{2}}\right] \exp\left[-2\left(\tau_2^2 - \tau_1^2\right)^{\frac{1}{2}}\right]\right\} \Big/ \left(4\tau_2^2\right) \tag{9j}$$

$P_m(x)$ is the Legendre polynomial of order m and argument x, $a_0 = 1$ and $a_1 = a^*$, where a^* is the linear anisotropic scattering coefficient, which can be calculated using the relation[9]

$$a^* = \sum_{m=0}^{\infty} \frac{(-1)^m a_{2m+1}(2m)!}{2^{2m} m!(m+1)!} \tag{10}$$

where a_n are Legendre polynomial expansion coefficients of the scattering phase function $p(\mu, \mu')$ and are calculated in terms of the refractive index n and the size parameter $z[\pi d/\lambda$, where d is particle diameter and λ is the wavelength of the radiation of interest] of the medium.[8]

The radiative transfer problem will be solved here using a variational technique[6] whereas the nonlinear relationship for the energy source term will be solved using an iterative method.

Variational Method

To solve the radiative transfer problem described by Eq. (8) using the variational technique,[6] the operators must be symmetric. Thus, multiplying Eq. (8) by $\sqrt{[\omega(\tau)]}$ leads to

$$\hat{T}_m G_m^*(y) = \Phi_m(\tau) \tag{11}$$

where

$$\hat{T}_m = \int_{\tau_1}^{\tau_2} \mathrm{d}y \left[\delta(\tau - y) + \left(\frac{a_m}{2}\right) L_{mm}^*(\tau, y)\right] \tag{12a}$$

$$G_m^*(\tau) = \tau\sqrt{\omega(\tau)} G_m(\tau) \tag{12b}$$

$$L_{mn}^*(\tau, y) = \sqrt{\omega(\tau)} L_{mn}(\tau, y) \sqrt{\omega(y)} \tag{12c}$$

$$\Phi_m(\tau) = \sqrt{\omega(\tau)} H_m(\tau) + \left(\frac{a_n}{2}\right) \int_{\tau_1}^{\tau_2} \mathrm{d}y G_m^*(y) L_{mn}(\tau, y) + \sqrt{\omega(\tau)} \int_{\tau_1}^{\tau_2} \mathrm{d}y [1 - \omega(y)] y \Theta^4(y) L_{m0}(\tau, y), \qquad n \neq m \tag{12d}$$

The solution using the variational method is carried out by assuming trial functions $\bar{G}_0(\tau)$ and $\bar{G}_1(\tau)$, which tend to converge to $G_0^*(\tau)$ and $G_1^*(\tau)$ after optimizing the two functionals defined by

$$F_m(\bar{G}_m) = 2\int_{\tau_1}^{\tau_2} \mathrm{d}\tau \bar{G}_m(\tau) \Phi_m(\tau) - \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \bar{G}_m(\tau) \hat{T}_m \bar{G}_m(y) \qquad m = 0, 1 \tag{13}$$

The two trial functions are assumed to be in terms of the asymptotic solution of the problem as[6,7]

$$\bar{G}_m(\tau) = \left[\tau/\sqrt{\omega(\tau)}\right][A_{1m} \exp(-\nu\tau) + A_{2m} \exp(\nu\tau)] \tag{14}$$

where A_{1m} and A_{2m} are constants to be determined whereas ν is defined by the transcendental equation

$$(2\nu/\bar{\omega})[\nu^2 + a^*\bar{\omega}(1 - \bar{\omega})]/[\nu^2 + a^*(1 - \bar{\omega})] = \ln[(1 + \nu)/(1 - \nu)] \tag{15}$$

where $\bar{\omega}$ is the spatial average of $\omega(\tau)$; that is,

$$\bar{\omega} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \tau^2 \omega(\tau) \Bigg/ \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \tau^2$$

The constants A_{1m} and A_{2m} are determined by optimizing the functionals F_m as

$$\frac{\partial F_m}{\partial A_{1m}} = \frac{\partial F_m}{\partial A_{2m}} = 0, \qquad m = 0, 1 \tag{16}$$

This optimization leads to use the trial functions $\bar{G}_m(\tau)$ instead of $G_m^*(\tau)$ to calculate $G_m(\tau)$.

Calculations Method

To solve the radiative transfer equation, we must know the dimensionless temperature distribution $\Theta(\tau)$. The dimensionless temperature distribution is described by the conductive energy equation that contains the dimensionless radiative net flux that is given by solving the radiative transfer equation. Because the relation between the dimensionless temperature distribution and the dimensionless radiative net flux is a nonlinear relation, we will use an iterative method to solve the problem. The iteration procedure is started assuming a dimensionless temperature distribution

$$\Theta(\tau) = \Theta_2 - [(\tau_2 - \tau)/(\tau\tau_2)][(\tau_1\tau_2)/(\tau_2 - \tau_1)] \times \left[(\Theta_2 - \Theta_1) + (H/6)\left(\tau_2^2 - \tau_1^2\right)\right] + (H/6)\left(\tau_2^2 - \tau^2\right) \tag{17}$$

This form of $\Theta(\tau)$ is used in Eq. (8) with the variational method to calculate the first iteration of the dimensionless radiative net flux $G_1(\tau)$. These values of the dimensionless radiative net flux are used in Eq. (7) to find new values of $\Theta(\tau)$, which are used in Eq. (8) to calculate new values of $G_1(\tau)$, and so on, to yield convergent results for both $\Theta(\tau)$ and $G_1(\tau)$.

The conductive–radiative physical quantities that will be calculated here are the dimensionless temperature distribution $\Theta(\tau)$ defined by Eq. (7) and the dimensionless total heat flux, which is the sum of the dimensionless conductive heat flux and the dimensionless radiative heat flux, defined by

$$Q_T(\tau) = -\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau} + \frac{G_1(\tau)}{(2N)} \tag{18}$$

Results and Discussion

The dimensionless temperature distribution $\Theta(\tau)$ and dimensionless total heat flux $Q_T(\tau)$ are calculated for a spherical shell of radii $\tau_1 = 1$ and $\tau_2 = 2$ and for boundaries of constant temperatures $\Theta_1 = 1$ and $\Theta_2 = 0$. The calculations are carried out for isotropically scattering, and for forward and backward anisotropically scattering, inhomogeneous media.

The effects of the conduction–radiation parameter N and the heat generation H on the temperature and total heat

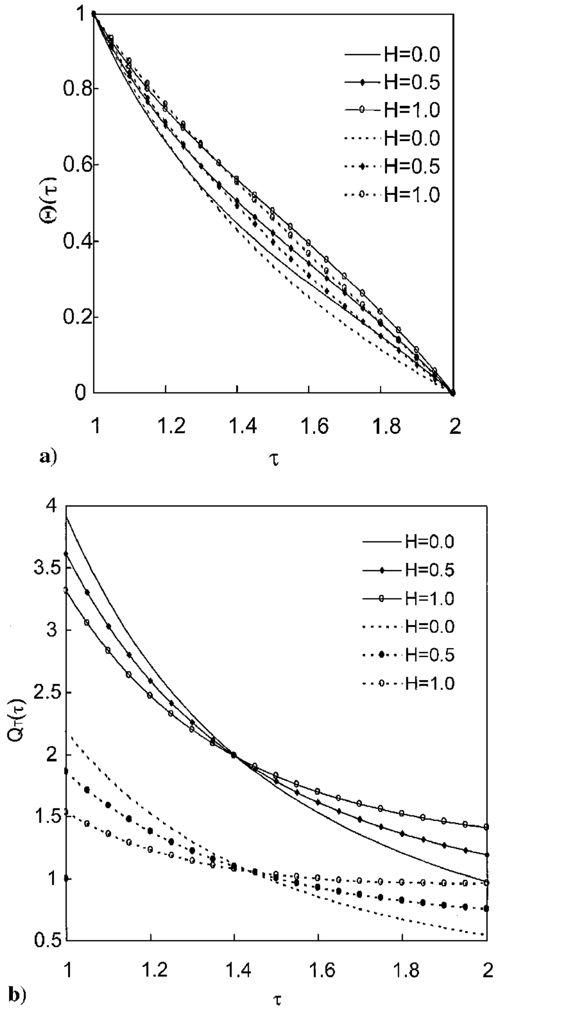

Fig. 1 Effects of conduction–radiation parameter and heat generation on distributions of a) temperature and b) total heat flux.

flux distributions for an inhomogeneous medium of isothermal, transparent boundaries with a single scattering albedo $\omega(\tau) = 0.8 - 0.4\tau/W_1$ for isotropic scattering are studied in Fig. 1. Here, $W_n = (\tau_2^{n+3} - \tau_1^{n+3})/(\tau_2^3 - \tau_1^3)$. The solid curves are for $N = 0.1$ and the dotted curves for $N = 1.0$. The results show that the conduction–radiation parameter has a small effect on the temperature distribution inside the medium. The total heat flux is strongly affected by the conduction-radiation parameter where it decreases as the parameter increases. The temperature inside the medium increases as H increases. The total heat flux decreases in the medium from the inner surface to the center, whereas it increases from the center to the outer surface as H increases.

In Fig. 2, the calculations of the dimensionless temperature and total heat flux distributions are carried out for inhomogeneous media having conduction–radiation parameter $N = 0.1$, heat generation $H = 0.5$, and isothermal, reflecting boundaries with diffuse reflectivities $\rho_1^d = \rho_2^d = 0.2$. The calculations are carried out for forward (solid curves) and backward (dotted curves) anisotropic scattering using anisotropic coefficients $a^* = 1.81715$ and $a^* = -0.58659$, respectively. The forward anisotropic scattering medium has refractive index $n = 1.2$ and size parameter $z = 2$, and the backward anisotropic medium has $n = \infty$ and $z = 1$. The coefficients a_m in Eq. (10) for both cases are taken from Özişik.[8] The results are given for single scattering albedos $\omega 1(\tau) = 2\tau/15W_1$, $\omega 2(\tau) = 0.8 - 0.4\tau/W_1$, and $\omega 3(\tau) = 0.8 + 0.5\tau/15W_1 - 0.4\tau^2/W_2$, which have average values 0.1, 0.5, and 0.8, respectively. The calculations show that increasing of the single scattering albedo decreases the temperature inside the medium while the total heat flux is almost unaffected. Also, the temperature distribution inside the medium is relatively insensitive to the change in the anisotropic scattering parameter while the total heat flux decreases as the anisotropic parameter decreases. However, the effects of both the single scattering albedo and the anisotropic parameter are small.

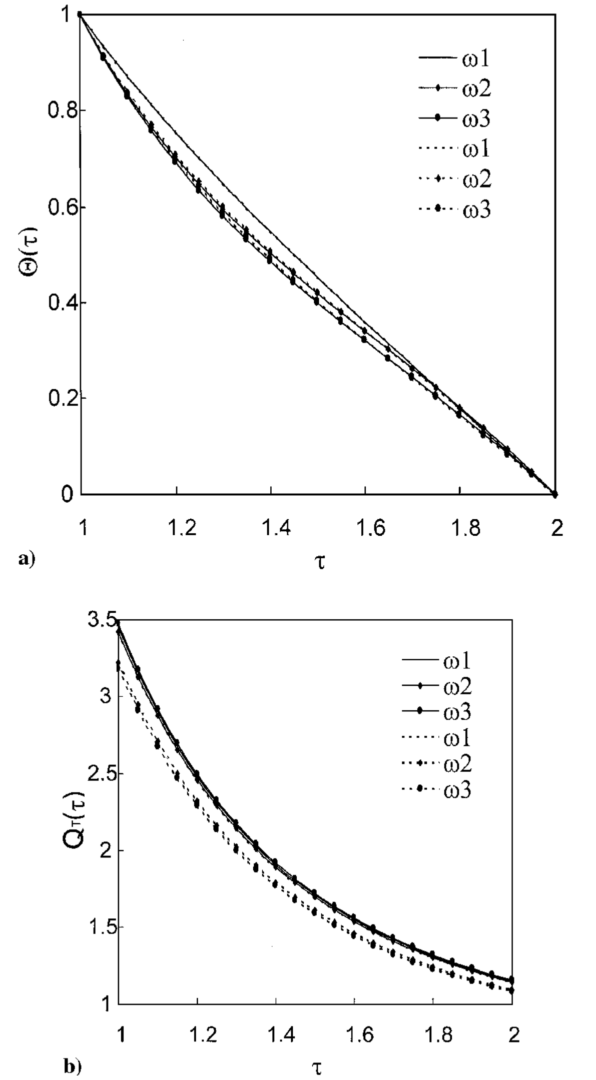

Fig. 2 Effects of the single scattering albedo and the anisotropic scattering coefficient on distributions of a) temperature and b) total heat flux.

Conclusions

The coupled of conductive and radiative heat transfer in an inhomogeneous, anisotropically scattering gray medium contained in a spherical shell of diffusely reflecting boundaries having constant temperatures is considered. The variational method is used to solve the radiative problem using trial functions in terms of the Pomraning solution[7] (also see Ref. 6) of the differential form of the free-source problem. The nonlinearity problem between the radiative transfer equation and the conductive energy equation is solved using an iterative method. The iteration procedure is started assuming a first temperature distribution, which is used to calculate the first iteration of the radiative net flux. These values of the radiative net flux are used to find new values for the temperature distribution, which are used to calculate new values of the radiative net flux, and so on, to yield convergent results for both the temperature distribution and the radiative net flux.

The calculations are carried out for inhomogeneous media with different forms of spatially dependent single scattering albedo. The results are given for forward anisotropic, isotropic, and backward anisotropic media. The boundaries of the media are considered as transparent or diffusely reflecting boundaries with constant temperatures. These calculations are used to study the effects of the single scattering albedo, the anisotropic parameter, the conduction–radiation parameter, and the heat source on the physical quantities. The single scattering albedo and the anisotropic scattering coefficient have small effects on the physical quantities whereas the conduction–radiation parameter and the heat generation have greater effects.

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